

HALL CURRENT EFFECTS IN MEAN-FIELD DYNAMO THEORY

MANASVI LINGAM

Department of Astrophysical Sciences and Princeton Plasma Physics Laboratory, Princeton University, Princeton, NJ 08544, USA

AMITAVA BHATTACHARJEE

Department of Astrophysical Sciences and Princeton Plasma Physics Laboratory, Princeton University, Princeton, NJ 08544, USA
 Max Planck-Princeton Center for Plasma Physics, Princeton University, Princeton, NJ 08544, USA

ABSTRACT

The role of the Hall term on large scale dynamo action is investigated by means of the First Order Smoothing Approximation. It is shown that the standard α coefficient is altered, and is zero when a specific double Beltrami state is attained, in contrast to the Alfvénic state for MHD dynamos. The β coefficient is no longer positive definite, and thereby enables dynamo action even if α -quenching were to operate. The similarities and differences with the (magnetic) shear-current effect are pointed out, and a mechanism that may be potentially responsible for $\beta < 0$ is advanced. The results are compared against previous studies, and their astrophysical relevance is also highlighted.

1. INTRODUCTION

The importance of large scale magnetic fields in the Universe, ranging from the cosmic to the planetary, cannot be overstated (Kulsrud & Zweibel 2008; Subramanian 2016). They play a chief role in processes such as star formation, galaxy evolution, etc. and as a means of probing and interpreting astrophysical environments. The principal process responsible for generating these large scale fields is the dynamo mechanism, which has been extensively studied for nearly a century (Krause & Rädler 1980; Yoshizawa et al. 2004; Brandenburg & Subramanian 2005), but there are several key issues that are yet unresolved (Brandenburg et al. 2012).

To date, most analyses in plasma astrophysics, and, by extension, in dynamos, rely on ideal or resistive magnetohydrodynamics (MHD) as the underlying physical model on account of its simplicity. However, the MHD approximation is *not* universally valid, and there are many domains where other plasma effects become dominant. One of the simplest of these ‘extended’ models is Hall MHD, in which the ions and electrons generally move with different flow velocities. This effect is manifested in the Ohm’s law via the Hall term, and the relative magnitude of such a term is typically encoded in the Hall parameter - the ratio of the ion skin depth to the characteristic background scale length.

In recent times, the Hall parameter has been shown to be ‘large’ in diverse environments such as protoplanetary discs (Wardle 2007), the ISM, the Earth’s magnetotail (Bhattacharjee 2004), the solar corona and the solar wind (Bhattacharjee 2004; Galtier & Buchlin 2007) to name a few. Furthermore, it is also likely to be highly relevant in space and laboratory settings (Huba 1995; Bhattacharjee et al. 2001), especially in experiments involving the Reversed Field Pinch (Ji et al. 1994; Ji 1999; Ding et al. 2004). In each of these instances, understanding the dynamics of the magnetic fields would arguably necessitate the use of Hall MHD, as opposed to MHD, as the base physical model.

Despite the ubiquity and importance of the Hall term, dynamo models that incorporate this term have been few and far between. The notable exceptions in large scale dynamo theory are the series of works undertaken by Mininni and collaborators (Mininni et al. 2002, 2003, 2005, 2007) as well as a few other publications (Ji 1999; Lingam & Mahajan 2015). There have also been some concomitant studies of small-scale Hall MHD dynamos, such as Kleorin & Rogachevskii (1994); Mirnov et al. (2003); Gómez et al. (2010). The Hall MRI dynamo has been subjected to detailed investigations only in recent times (Sano & Stone 2002; Kunz & Lesur 2013; Lingam & Bhattacharjee 2016).

In this paper, we shall use incompressible Hall MHD and carry out an investigation along the lines of Gruzinov & Diamond (1994) and Gruzinov & Diamond (1995), which will henceforth be referred to as GD94

and GD95 respectively. At this stage, a crucial comment pertaining to our approach is necessary. Our model does not include stochasticity and chaos (Vishniac & Brandenburg 1997; Bushby & Tobias 2007), the shear-current effect (Rogachevskii & Kleeorin 2003, 2004; Squire & Bhattacharjee 2016), helicity flux contributions (Vishniac & Cho 2001; Subramanian & Brandenburg 2004; Sur et al. 2007; Ebrahimi & Bhattacharjee 2014), dynamical equations for the large- and small-scale helicities (Blackman & Field 2002; Brandenburg et al. 2012; Blackman 2015), and hyperresistivity (Bhattacharjee & Yuan 1995; Brandenburg & Subramanian 2005) to name a few. Although each and every one of these effects is undoubtedly important, the central goal of the paper is to understand how the Hall term acts in near-isolation.

We shall show that the Hall term introduces several non-trivial effects, such as the existence of a non-positive definite diffusion coefficient, the drive towards a non-Alfvénic final state, and the possibility for dynamo action despite α -quenching (Vainshtein & Cattaneo 1992; Gruzinov & Diamond 1994; Cattaneo & Hughes 1996) amongst others. Collectively, we argue that these non-trivial effects make a compelling case for carrying out in-depth Hall MHD dynamo analyses in the future.

2. MATHEMATICAL PRELIMINARIES AND THE KINEMATIC PICTURE

We briefly outline the mathematical equations of our model, and present a kinematic derivation of the dynamo coefficients in this Section.

2.1. The governing equations of the model

Let us commence our analysis by presenting the equations for incompressible Hall MHD wherein the density obeys the relation $\rho = \text{const}$. The dynamical equation for the velocity is

$$\left[\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right] \mathbf{V} = (\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla \left(\frac{p}{\rho} \right), \quad (1)$$

and the Ohm's law can be expressed as

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V}_E \times \mathbf{B}) - \eta \nabla \times \mathbf{J}, \quad (2)$$

where $\mathbf{J} = \nabla \times \mathbf{B}$, $\mathbf{V}_E = \mathbf{V} - d_i \mathbf{J}$, η is the resistivity, and d_i is the ion skin depth. Also note that these equations have been expressed in Alfvénic units where \mathbf{B} has the units of (Alfvén) velocity. The above set of equations must be supplemented with $\nabla \cdot \mathbf{V} = \nabla \cdot \mathbf{B} = 0$.

Before proceeding further, an interesting mathematical property is worth emphasizing. If we let $\mathbf{B} \rightarrow -\mathbf{B}$ in (2), we find that the equation is *not* invariant under this transformation. In other words, it is the Hall term

(that is linear in d_i) that breaks this symmetry. Hence, we may expect all terms linear in d_i that appear in the electromotive force to violate this symmetry - we shall establish the validity of this conjecture in Sec. 3.1.¹

2.2. A kinematic study of the Hall dynamo

To maintain consistency with GD94 and GD95, we adopt the same properties of the velocity field, i.e. the large scale component is taken to be negligible. This is, of course, quite a strong simplification since the inclusion of large scale velocity with shear leads to the “shear-current” effect (Rogachevskii & Kleeorin 2003, 2004; Squire & Bhattacharjee 2016). We split the magnetic field into the mean field \mathbf{B}_0 , and the small-scale component \mathbf{b} . The mean field evolves via

$$\frac{\partial \mathbf{B}_0}{\partial t} = \nabla \times \mathcal{E} - d_i \nabla \times (\mathbf{J}_0 \times \mathbf{B}_0) - \eta \nabla \times \mathbf{J}_0, \quad (3)$$

where the electromotive force \mathcal{E} is determined through

$$\mathcal{E} = \langle (\mathbf{v} - d_i \nabla \times \mathbf{b}) \times \mathbf{b} \rangle, \quad (4)$$

and $\langle \dots \rangle$ denotes the ensemble average (over the turbulent field \mathbf{v}). Upon applying the well-known first order smoothing approximation (FOSA) from kinematic dynamo theory (Krause & Rädler 1980; Brandenburg & Subramanian 2005), we find that the turbulent component of the magnetic field obeys

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times [(\mathbf{v} - d_i \mathbf{j}) \times \mathbf{B}_0] - d_i \nabla \times (\mathbf{J}_0 \times \mathbf{b}). \quad (5)$$

In obtaining the above relation, note that the resistive term has been neglected, as it does not play a major role in our subsequent analysis; for more details, refer to Section 6.3.1 of Brandenburg & Subramanian (2005). Now, we can substitute (5) into (4) and implement the Reynolds relations, and the assumptions of statistical homogeneity and isotropy (Krause & Rädler 1980; Brandenburg & Subramanian 2005). The latter amounts to replacing correlation tensors (for turbulent quantities) of rank 2 and 3 with δ_{ij} and ϵ_{ijk} respectively. Upon doing so, we find that (4) simplifies to

$$\mathcal{E} = \alpha \mathbf{B}_0 - \beta \nabla \times \mathbf{B}_0 + \dots, \quad (6)$$

where the ‘ \dots ’ indicate the existence of higher-order derivative terms in \mathbf{B}_0 .

Let us now turn our attention to (4) in order to understand how it must be computed. We observe that it may be written as

$$\begin{aligned} \mathcal{E} = \tau_c \Big[& \langle \partial_t \mathbf{v} \times \mathbf{b} \rangle + \langle \mathbf{v} \times \partial_t \mathbf{b} \rangle - d_i \langle (\nabla \times \mathbf{b}) \times \partial_t \mathbf{b} \rangle \\ & - d_i \langle (\nabla \times \partial_t \mathbf{b}) \times \mathbf{b} \rangle \Big], \end{aligned} \quad (7)$$

¹ The value of such a symmetry analysis has already been well established in gauge theories (Weinberg 1976; Jackson 1999).

upon invoking the correlation time approximation (Brandenburg & Subramanian 2005); in the above expression, τ_c denotes the correlation time. It is easy to verify that the second and third terms in the above expression can be clubbed together as $\langle \mathbf{v}_E \times \partial_t \mathbf{b} \rangle$. We have dubbed the last three terms of (7) as ‘kinematic’ since they depend solely on the evolution of $\partial_t \mathbf{b}$, and are thus analogous to the standard kinematic treatments of the MHD dynamo (Brandenburg & Subramanian 2005). It is, however, important to recognize that there *are* non-linear effects arising from the Hall drift, but these terms are not reliant upon the momentum equation (1). On the other hand, the first term in (7) is markedly different as it explicitly depends upon the temporal evolution of \mathbf{v} . For this reason, the first term is tackled in greater detail in the next section.

We shall concern ourselves presently with the kinematic picture by evaluating the last three terms in (7). We find that the coefficients occurring in (6) are thus given by

$$\alpha_0 = -\frac{\tau_c}{3} \langle \mathbf{v}_E \cdot (\nabla \times \mathbf{v}_E) + d_i \mathbf{b} \cdot (\nabla \times \nabla \times \mathbf{v}_E) \rangle, \quad (8)$$

$$\beta_0 = \frac{\tau_c}{3} \langle \mathbf{v}_E^2 + d_i (\mathbf{b} \cdot \nabla \times \mathbf{v}_E + \mathbf{v}_E \cdot \nabla \times \mathbf{b}) + d_i^2 \mathbf{b} \cdot \nabla \times \nabla \times \mathbf{b} \rangle, \quad (9)$$

where the subscript ‘0’ associated with the coefficients indicates that they represent the kinematic values. In (8) and (9), $\mathbf{v}_E = \mathbf{v} - d_i \nabla \times \mathbf{b}$ represents the (turbulent) electron velocity. The first and second terms in the α -coefficient, whose explicit expression can be found in (8), arise from $\langle \mathbf{v}_E \times \partial_t \mathbf{b} \rangle$ and $\langle (\nabla \times \partial_t \mathbf{b}) \times \mathbf{b} \rangle$ respectively by means of (4). Similarly, the first and second terms in (9) originate via $\langle \mathbf{v}_E \times \partial_t \mathbf{b} \rangle$, and the last two terms in (9) emerge from $\langle (\nabla \times \partial_t \mathbf{b}) \times \mathbf{b} \rangle$.

A few comments are in order. Firstly, we note that the *dimensional* limit $d_i \rightarrow 0$ transforms (8) and (9) to the standard α and β contributions from MHD kinematic dynamo theory (Krause & Rädler 1980). Secondly, an inspection of (9) reveals that β_0 is no longer positive definite - a feature that we explore in greater detail in Sec. 4. Lastly, it is also possible to compute higher order contributions that appear in (6). The so-called γ -effect (Mininni et al. 2007) is existent when \mathcal{E} contains terms that are proportional to $\nabla \times (\nabla \times \mathbf{B}_0)$. Upon implementing the same procedure, we find that

$$\gamma_0 = -\frac{\tau_c}{3} d_i \langle \mathbf{v}_E \cdot \mathbf{b} \rangle, \quad (10)$$

which clearly vanishes in the MHD limit, i.e. upon taking $d_i \rightarrow 0$.

3. BEYOND THE KINEMATIC PARADIGM

We tackle the nonlinear picture in this Section, and briefly outline alternative approaches to our methodology in conjunction with the accompanying results.

3.1. The nonlinear regime of the Hall dynamo

To tackle the ‘nonlinear’ picture, we need to investigate the first term of (7), as pointed out in Sec. 2.2. This term allows us to account for the back-reactions arising from the Lorentz force in the momentum equation.

At this stage, we point out a remarkable fact. Since the first term is given by $\langle \partial_t \mathbf{v} \times \mathbf{b} \rangle$, it can involve Hall corrections only through the quantity $\partial_t \mathbf{v}$ since there are no other factors of d_i prevalent. However, it is well known that the evolution equation for the velocity (1) is exactly *identical* to its ideal MHD counterpart (Huba 1995). As a result, upon utilizing FOSA and the other assumptions of our model, it becomes Eq. (5) of GD94; see also Eq. (25) of GD95. The expression is given by

$$\partial_t \mathbf{v} = (\mathbf{B}_0 \cdot \nabla) \mathbf{b} + (\mathbf{b} \cdot \nabla) \mathbf{B}_0 - \nabla \left(\frac{p}{\rho} \right), \quad (11)$$

and it turns out that the α correction arises only from the first term on the RHS in the above equation. This is easy to verify after some careful inspection, or by following through the analysis in GD94 and GD95. Upon plugging the first term on the RHS of (11) into the first term on the RHS of (7), we obtain the nonlinear correction. It turns out to be proportional to $\langle \mathbf{b} \cdot \nabla \times \mathbf{b} \rangle$, and thus the final expression for α is given by

$$\alpha = -\frac{\tau_c}{3} \langle \mathbf{v}_E \cdot (\nabla \times \mathbf{v}_E) + d_i \mathbf{b} \cdot (\nabla \times \nabla \times \mathbf{v}_E) \rangle + \frac{\tau_c}{3} \langle \mathbf{b} \cdot (\nabla \times \mathbf{b}) \rangle, \quad (12)$$

and the term on the second line of the RHS is the nonlinear contribution arising from the back-reaction of the Lorentz force. As before, the limit $d_i \rightarrow 0$ leads to the classical nonlinear α coefficient first derived in Pouquet et al. (1976). It is instructive to compute how the nonlinear value of α differs from the kinematic value. In order to do so, we mirror the approach employed in GD94 and GD95. We write down the expression for the evolution of the vector potential (in a suitable gauge):

$$\partial_t \mathbf{a} = \mathbf{v}_E \times \mathbf{B}_0 + \mathbf{v}_E \times \mathbf{b} - \mathcal{E} - d_i \mathbf{J}_0 \times \mathbf{b} - \eta \nabla \times \mathbf{b}, \quad (13)$$

where $\mathbf{J}_0 = \nabla \times \mathbf{B}_0$ and \mathcal{E} was defined in (4). We take the dot product of the above equation with \mathbf{b} and then perform the ensemble averaging. To simplify the resultant relation, we invoke the helicity balance (based on small-scale stationarity) argument of GD94 and GD95, and use the relations $\langle \mathbf{b} \rangle = 0$ and $\mathbf{b} \cdot (\mathbf{b} \times \mathbf{X}) = 0$. We note that the above steps are justified since magnetic helicity is an invariant of Hall MHD, just as in ideal MHD (Turner 1986; Lingam et al. 2015, 2016). Before proceeding further, we emphasize that the results

of GD94 and GD95 are not truly universal, as they have been generalized by several authors - see for e.g. the work of [Bhattacharjee & Yuan \(1995\)](#) on hyperresistivity and [Blackman & Field \(1999\)](#); [Blackman \(2003\)](#); [Brandenburg & Subramanian \(2005\)](#).

After applying the aforementioned relations to (13) and carrying out the requisite algebra, we end up with

$$\langle \mathbf{b} \cdot (\mathbf{v}_E \times \mathbf{B}_0) \rangle = \eta \langle \mathbf{b} \cdot \nabla \times \mathbf{b} \rangle, \quad (14)$$

and the LHS is further simplified by noting that $\langle \mathbf{b} \cdot (\mathbf{v}_E \times \mathbf{B}_0) \rangle = -\mathbf{B}_0 \cdot \langle \mathbf{v}_E \times \mathbf{b} \rangle = -\mathbf{B}_0 \cdot \boldsymbol{\mathcal{E}}$. The last equality followed upon using the definition of \mathbf{v}_E in conjunction with (4). Thus, we recover

$$\langle \mathbf{b} \cdot \nabla \times \mathbf{b} \rangle = -\frac{\alpha}{\eta} B_0^2 + \frac{\beta}{\eta} \mathbf{B}_0 \cdot \nabla \times \mathbf{B}_0. \quad (15)$$

Upon substituting (15) into (12), we find that

$$\alpha = \left(\alpha_0 + \frac{\beta}{\eta} \mathbf{B}_0 \cdot \nabla \times \mathbf{B}_0 \right) \left(1 + \frac{\tau_c B_0^2}{3\eta} \right)^{-1}, \quad (16)$$

and it is possible to normalize the above equation in terms of the ‘‘Zel’dovich’’ units to obtain a result exactly analogous to GD94 and GD95. The only difference (thus far) is that the value of α_0 is given by (8), which is different from its MHD counterpart. However, note that our analysis is not fully complete since the quantity β in (16) involves potential nonlinear contributions, which are yet to be evaluated.

Fortunately, we can take advantage of the aforementioned fact that the velocity evolution equation is the same for Hall and ideal MHD. We shall not present the full details of our calculation, and instead sketch the essential steps. The Fourier representation of the velocity and magnetic field evolution equations is considered, analogous to Appendix B in GD95. The second term in Eq. (B2) is the crucial quantity, as it represents the nonlinear contributions. As it involves $\partial_t \mathbf{v}$, the vertex $\Gamma_{\alpha\beta\gamma}$ in Eq. (B3) is identical, because of the above equivalence between Hall and ideal MHD. Consequently, the rest of the algebra, and the ensuing result, remains the same. Hence, we are led to the conclusion that the nonlinear contributions, in three dimensions, to the β effect are nil. As a result, we can replace β in (16) with β_0 , where the latter is given by (9).

Let us turn our attention to (12) and (9) and perform the transformation $\mathbf{b} \rightarrow -\mathbf{b}$. We find that all terms linear in d_i are rendered non-invariant, i.e. they change parity, thereby confirming the prediction stated in Sec. 2.1, and the value of this analysis.

3.2. Alternative approaches to handling the Hall term

We commence by observing that (9) and (12) represent our final dynamo coefficients. They have been

obtained through a procedure analogous to the one outlined in GD94 and GD95. We also carried out the analysis for the Hall MHD large-scale dynamo using an alternative approach - a slightly modified version of the Minimal Tau Approximation (MTA) ([Blackman & Field 2002](#); [Rädler & Rheinhardt 2007](#)). We find that the same results, namely (9) and (12), are recovered.

As noted in Sec. 1, Hall MHD large scale dynamos have not been studied widely in the literature. On the theoretical front, [Mininni et al. \(2002, 2003\)](#) relied upon a variant of the Reduced Smoothing Approximation (RSA) ([Blackman & Field 1999](#)) to compute the α -coefficient. We find that our result is in exact agreement with their work. The β -coefficient was computed later by [Mininni et al. \(2007\)](#) using the same approach. We have verified that the two expressions for β , namely (9) and Eq. (3.10) of [Mininni et al. \(2007\)](#), are equivalent upon integration by parts, up to an overall divergence term that is proportional to $\nabla \cdot \langle \mathbf{b} \times (\nabla \times \mathbf{b}) \rangle$. It can vanish in certain scenarios, for e.g. when (i) periodic boundary conditions exist, or (ii) the current exhibits a sufficiently steep decline and becomes zero at infinity (in the case of spatial averaging).

4. ON THE IMPLICATIONS OF HALL MHD FOR LARGE SCALE DYNAMOS

In this Section, we shall address the implications of the Hall term on large scale dynamos, and indicate the environments wherein such effects are likely to be important.

4.1. The effect of the Hall term on the alpha coefficient

We shall begin our study by considering the nonlinear expression for α , which is given by (12).

It has been argued that nonlinear Alfvén waves serve as the ‘building’ blocks of plasma turbulence ([Biskamp 2003](#); [Howes & Nielson 2013](#)). In MHD, these waves obey the relation $\mathbf{b} = \pm \mathbf{v}$. In addition to the important property of equipartition, the Alfvénic solution has a crucial implication for large-scale dynamo theory - the α -effect attains a null value ([Pouquet et al. 1976](#); [Gruzinov & Diamond 1994](#)). The importance of Alfvénic states is further underscored by the fact that they are the asymptotic outcomes of MHD turbulence ([Dobrowolny et al. 1980](#); [Grappin et al. 1982](#); [Boldyrev 2006](#)).

On the other hand, when we consider our dynamo model, we find that Alfvénic fluctuations do *not* lead to $\alpha = 0$ due to the modifications induced by the Hall term. This condition is realized only when a very special ansatz is chosen - the famous double Beltrami states of Hall MHD ([Mahajan & Yoshida 1998](#)). It is obtained by extremizing the energy $E = \frac{1}{2} \int_D (v^2 + b^2) d^3x$ subject to the constraint that the two helicities

1. $H_1 = \int_D \mathbf{a} \cdot \mathbf{b} d^3x$, which denotes the usual magnetic helicity
2. $H_2 = \int_D (\mathbf{a} + d_i \mathbf{v}) \cdot (\mathbf{b} + d_i \nabla \times \mathbf{v}) d^3x$, which is often referred to as the canonical, or generalized, helicity (Turner 1986; Lingam et al. 2015, 2016)

are held constant. In mathematical terms, it is expressed via the variational principle $\delta F = 0$, where $F := E - 1/2(\lambda_1 H_1 + \lambda_2 H_2)$ is the relevant functional (Turner 1986; Mahajan & Yoshida 1998). Upon simplification, the double Beltrami states of Hall MHD are given by

$$\begin{aligned} \mathbf{v} &= d_i \lambda_2 (\mathbf{b} + d_i \nabla \times \mathbf{v}), \\ \nabla \times \mathbf{b} &= \lambda_1 \mathbf{b} + \lambda_2 (\mathbf{b} + d_i \nabla \times \mathbf{v}). \end{aligned} \quad (17)$$

For the multi-Beltrami states, it was also shown that $F = 0$ upon subsequent simplification (Mahajan & Lingam 2015). Thus, the resultant condition for $\alpha = 0$ is given by $\lambda_1 - \lambda_2 - 2\lambda_2 \lambda_1^2 d_i^2 = 0$. As the condition $F = 0$ is also applicable for the double Beltrami states, we have two equations for λ_1 and λ_2 . Hence, one can fully determine the precise values of λ_1 and λ_2 , in terms of E , H_1 , H_2 and d_i when the α -coefficient vanishes. For this specific double Beltrami state, i.e. when α is zero, we note that the condition $\beta > 0$ holds true.

In general, note that the double Beltrami solution has two free parameters. Owing to this freedom, it is possible for α to acquire positive, negative, or null values. This fact was also pointed out in Mininni et al. (2002), but it is important to recognize that the fluctuations do not necessarily need to exhibit a double Beltrami behaviour. However, it is possible to verify that the system must attain a *specific* double Beltrami state for the α -coefficient to vanish.

We have discussed the connection between Alfvén wave turbulence and the vanishing of the α -coefficient in MHD. Next, we have considered Hall MHD and shown that a particular double Beltrami state is responsible for the vanishing of α . Hence, it is natural to conjecture that the nonlinear Alfvén wave solutions of Hall MHD must be related (or can be constructed) from the double Beltrami states. This would ensure that Alfvén wave turbulence and large scale dynamo theory are equally connected in Hall MHD (as in MHD). We note that this intuitive picture is indeed correct, as the validity of the above conjecture was established in Mahajan & Krishan (2005); see also Appendix B of Lingam & Bhattacharjee (2016) for a discussion of the same. In mathematical terms, the Hall MHD nonlinear Alfvén waves obey the pair of relations

$$\begin{aligned} \mathbf{v} - \nabla \times \mathbf{b} &= \alpha \mathbf{b}, \\ \mathbf{b} + \nabla \times \mathbf{v} &= \frac{1}{\alpha} \mathbf{v}, \end{aligned} \quad (18)$$

which, upon suitable manipulation, can be reduced to a particular case of (17). The details of the algebra behind (18) can be found in Sec. II of Mahajan & Krishan (2005). Lastly, we wish to emphasize that the double Beltrami state, which leads to a null value of the α -coefficient and shares connections with Hall MHD wave turbulence, is also accompanied by a lack of equipartition. The latter trait is quantified via the expression

$$\mathbf{b}_k = \alpha_{\pm} \mathbf{v}_k, \quad (19)$$

where the subscript ‘ k ’ denotes the Fourier component of the corresponding fields. Here, α_{\pm} is given by

$$\alpha_{\pm} = -\frac{k}{2} \pm \sqrt{\frac{k^2}{4} + 1}, \quad (20)$$

and the wave vector k has been normalized in units of d_i . An inspection of (19) reveals clear differences upon comparison with the MHD Alfvénic solution $\mathbf{b} = \pm \mathbf{v}$, and it is also readily apparent that there is a clear non-equipartition; the latter arises because of the k -dependence that leads to $b_k^2 \neq v_k^2$.

4.2. The effect of the Hall term on the beta coefficient

One of the important consequences of our work is that β is *not* positive definite, as opposed to its MHD counterpart. This feature has important consequences since it enables a non-local energy transfer from the small to large scales, as shown in the simulations by Mininni et al. (2007). The analytical work presented in Mininni et al. (2007) also establishes that $\beta < 0$ is feasible, and agrees with our findings.

The shear-current effect that was analyzed in detail by Rogachevskii & Kleeorin (2003, 2004) has sparked a fair degree of controversy and extensive further research (Rüdiger & Kitchatinov 2006; Brandenburg et al. 2008; Yousef et al. 2008; Hughes & Proctor 2009; Sridhar & Singh 2010), but one of the appealing features of shear is that the β effect can acquire negative off-diagonal contributions. The standard paradigm for the shear-current effect was recently revised by Squire & Bhattacharjee (2015, 2016), who also coined the phrase magnetic shear-current effect. Through a combination of numerical and analytic work, these authors confirmed the existence of a negative anisotropic diffusivity.

However, our work is very different as it functions in the *absence* of large-scale velocity shear, and we operate with a much simpler ansatz endowed with isotropy. It is important to recognize that the α -quenching occurs via (16), but β is not subject to such a restriction. Thus, the fact that $\beta < 0$ is permitted within our Hall MHD dynamo enables the mechanism to be functional, even when α may be quenched. For this reason, it is instructive to see our work as fundamentally *different*, but

also *complementary* to the above papers. In the latter scenarios, velocity shear is the contributing factor for negative diffusivity, whilst in our case, it is collisionless effects (the Hall term) that play an analogous role.

4.3. Where are the Hall effects important?

To gain an estimate of the new terms arising from the Hall term, let us focus on the α effect alone. Of the three terms in (12), only the second one is identical to the MHD case. Carrying out a naive dimensional analysis, we find that

$$\frac{|\mathbf{v}_E \cdot (\nabla \times \mathbf{v}_E)|}{|\mathbf{b} \cdot (\nabla \times \mathbf{b})|} \sim (\mathcal{M}_A \pm \epsilon_H)^2, \quad (21)$$

$$\frac{|\mathbf{b} \cdot (\nabla \times \nabla \times \mathbf{v}_E)|}{|\mathbf{b} \cdot (\nabla \times \mathbf{b})|} \sim \epsilon_H (\mathcal{M}_A \pm \epsilon_H), \quad (22)$$

where $\mathcal{M}_A = v/v_A$ is the turbulent Alfvén Mach number and $\epsilon_H = d_i/L$ is the Hall parameter, with L denoting the scale length. The ‘ \pm ’ has been inserted since the velocity and magnetic fluctuations may be parallel or anti-parallel. Allowing for $\epsilon_H \rightarrow 0$ leads us to the equivalent MHD scalings. Thus, the Hall effects become dominant when the Hall parameter is significant, which is entirely along expected lines. Some examples of systems wherein this parameter is large have been presented in Sec. 1.

5. DISCUSSION AND CONCLUSION

In this paper, we have investigated the role of the Hall effect on conventional large scale dynamo theory. For the purposes of physical simplicity, we have used a very simple ansatz, along the lines of [Gruzinov & Diamond \(1994, 1995\)](#), as the basis of our investigations. As discussed in Sec. 1, the calculations presented here are only the first step in a more complete treatment of the Hall dynamo, which will be extended in future work(s) to include several effects that have been omitted for simplicity. Foremost among them is the effect of large-scale velocity shear, which has led to the prediction of the novel “magnetic shear current effect” ([Squire & Bhattacharjee 2016](#)). We have chosen to work with such a model as it helps clarify the effects engendered by the Hall term. In ‘real’ systems, however, one would expect it to act in concert with all of the other phenomena alluded to in Sec. 1.

Our final results are encapsulated by (9), (12) and (16). Thus, we find that the α and β effects are modified when compared to their MHD counterparts, and that α -quenching operates analogous to GD94 and GD95 as well as [Vainshtein & Cattaneo \(1992\)](#); [Cattaneo & Hughes \(1996\)](#). Each of these results has important physical interpretations and consequences. We have also carried out a simple dimensional analysis in Sec. 4.3 and concluded that the dynamo coefficients

are significantly altered when the Hall parameter is sufficiently large, and the latter does occur in several space, astrophysical and laboratory plasmas, as pointed out in Sec. 1.

We observe that the null value of α is achieved via the process of ‘Beltramization’ in Hall MHD, i.e. when the system attains a particular double Beltrami state, as opposed to ‘Alfvénization’ ([Dobrowolny et al. 1980](#); [Grappin et al. 1982](#)) in MHD. We find that (i) wave turbulence (mediated via nonlinear Alfvén waves), (ii) relaxed states (the double Beltrami states can be thus interpreted), and (iii) large scale dynamo theory are intimately connected in Hall MHD. We believe that the connections between these three (highly important) areas of plasma physics certainly merit further investigation - a similar line of thought has also been advanced in a recent review by [Moffatt \(2016\)](#), albeit for MHD.

Moreover, there has been a great deal of interest in the diffusivity (β) becoming negative, and driving the large scale dynamo. Unlike many standard treatments that rely on large scale velocity shear or stochasticity ([Vishniac & Brandenburg 1997](#); [Rogachevskii & Kleeorin 2003](#); [Squire & Bhattacharjee 2016](#)), we have shown that the isotropic diffusion tensor can become negative under the influence of the Hall term, in concordance with earlier results ([Mininni et al. 2007](#)). We reiterate that the non-positive definiteness of β induced by the Hall term is likely to complement (and supplement) the shear-current effect, and not necessarily supplant it.

At this juncture, we proffer a potential explanation as to why $\beta < 0$ is possible. The backreactions of the Lorentz force ($\mathbf{J} \times \mathbf{B}$) are responsible for introducing the current helicity contribution to the α effect that is *opposite* in sign to the kinematic fluid helicity ([Pouquet et al. 1976](#); [Brandenburg & Subramanian 2005](#)). However, the same term is also present in the Ohm’s law of Hall MHD, and is responsible for new contributions to the dynamo coefficients - this is seen by inspecting the last two terms on the RHS of (7). It is precisely these new contributions that destroy the positive definite nature of β . Hence, in a manner of speaking, the $\mathbf{J} \times \mathbf{B}$ term in the Ohm’s law is solely responsible for generating new contributions that ‘oppose’ the tendency for $\beta > 0$. Thus, it is quite plausible that the $\mathbf{J} \times \mathbf{B}$ term modifies *both* the α and β coefficients and enforces this ‘opposite’ behaviour.

As the Hall term is important in laboratory and astrophysical environments, gives rise to important and subtle physical effects in large scale dynamos, and connects the latter field with other fundamental areas of plasma physics, we suggest that further studies of the kind carried out in the present paper are timely.

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